JOURNAL OF ALGEBRA 22, 409–427 (1972)

Semigroups of Order Preserving Partial Transformations of a Totally Ordered Set*

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Communicated by G. B. Preston

Received June 1, 1970

INTRODUCTION

If S is an inverse semigroup and we define the relation θ on the lattice, $\Lambda(S)$, of congruences on S by saying that two congruences are θ-equivalent if and only if they induce the same partition of the idempotents; then θ is a congruence relation on the lattice $\Lambda(S)$, and we write $\Theta(S)$ for $\Lambda(S)/\theta$. The relationship between $\Theta(S)$ and certain equivalence relations on a partially ordered set X, when S is an inverse semigroup of order preserving partial transformations of X was considered in [10]. Here we continue that study in the special case where X is totally ordered. We denote by $\mathcal{J}_X$ the subsemigroup of the symmetric inverse semigroup on X consisting of those elements α for which the domain, $\Delta(\alpha)$, and the range, $\nabla(\alpha)$, of α are ideals of X, and α is an order isomorphism of $\Delta(\alpha)$ onto $\nabla(\alpha)$. Let S be a subsemigroup of $\mathcal{J}_X$; then a convex equivalence relation ρ on X is a convex congruence if, for $x, y \in \Delta(\alpha)$, $\alpha \in S$, $(x\alpha, y\alpha) \in \rho$ if and only if $(x, y) \in \rho$. We call S o-primitive if the universal and identity convex congruences are the only convex congruences.

In the main theorem of Section 2, it is shown that if $|X| > 2$ and S is an o-primitive inverse subsemigroup of $\mathcal{J}_X$ then either X is anti-isomorphic to the natural numbers or isomorphic to the set of integers (in either of these cases S is transitive), or every transitivity class Y in the Dedekind completion $\bar{X}$ of X is dense in $\bar{X}$. (In certain circumstances, the converse will also hold.) The induced representation of S on any transitivity class is faithful and o-primitive.

The results in this section extend to certain semigroups of mappings results of Holland's and McCleary's in [3] and [4] with a view to applying them in Section 3 to $\Theta(S)$ for certain semigroups S.

* The research for this paper was supported, in part, by N.R.C. Grant No. A-4044

In Section 3 we combine the results of Section 2 and of [10] with Munn's representation (Lemma 3.1, [7]) of an inverse semigroup S on the semilattice of idempotents of S to obtain certain facts about $\Theta(S)$ for certain inverse semigroups S. For instance, if E_S is totally ordered, $|E_S| > 2$ and $|\Theta(S)| = 2$ then either S is a bisimple ω-semigroup or S is a bisimple Z-semigroup or each $\mathscr{D}$-class of idempotents is dense in E_S. Moreover, each $\mathscr{D}$-class D of S is a bisimple inverse semigroup such that E_D is totally ordered and $|\Theta(D)| = 2$.

I. Basic Results

We adopt the notation and terminology of [2]. In particular, a semigroup S is called an *inverse semigroup* if $a \in aSa$, for all $a \in S$, and the idempotents of S commute. Then there is a unique element x such that $a = axa$ and $x = xax$. We call x the *inverse* of a and write $x = a^{-1}$. For any inverse semigroup S, we denote E_S the subsemigroup of S consisting of the idempotents of S. There is a natural partial ordering on S, compatible with the multiplication, defined by $a \leqslant b$ if and only if $ab^{-1} = aa^{-1}$. With respect to this partial ordering E_S is a *semilattice* where, by a semilattice, we mean a partially ordered set in which any two elements x and y have a greatest lower bound, which we denote by $x \wedge y$. By a *lattice* we mean a semilattice in which any two elements also have a least upper bound, which we denote by $x \vee y$. For the basic results on inverse semigroups the reader is referred to [2].

Denote by $\Lambda(S)$ the lattice of congruences on an inverse semigroup S; that is, the lattice of equivalence relations ρ such that, for $a, b, c \in S$, $(a, b) \in \rho$ implies that $(ca, cb) \in \rho$ and $(ac, bc) \in \rho$.

Define the relation θ (cf. [11]) on $\Lambda(S)$ by $(\rho_1, \rho_2) \in \theta$ if and only if $\rho_1 \mid E_S = \rho_2 \mid E_S$, where $\rho_i \mid E_S$ denotes the restriction of the congruence ρ_i to E_S. Then

LEMMA 1.1 ([11] Theorem 5.1). *Let S be an inverse semigroup and the relation θ be defined as above. Then*

(i) *θ is a congruence on $\Lambda(S)$;*

(ii) *each θ-class is a complete modular sublattice of $\Lambda(S)$ (with a greatest and least element).*

We shall denote the lattice of θ-classes of an inverse semigroup S by $\Theta(S)$.

Now each congruence on an inverse semigroup S determines a *normal partition* of E_S; that is a partition $P = \{E_\alpha : \alpha \in J\}$ such that

E(i) $\alpha, \beta \in J$ implies that there exists a $\gamma \in J$ such that $E_\alpha E_\beta \subseteq E_\gamma$;

E(ii) $\alpha \in J$ and $a \in S$ implies that there exists a $\beta \in J$ such that $aE_\alpha a^{-1} \subseteq E_\beta$.

We call an equivalence relation ρ on E_S a *normal* equivalence if its classes constitute a normal partition of E_S.

Conversely, if P is a normal partition of E_S then P is induced by some congruence on S. Thus the lattice of normal partitions of E_S is just (isomorphic to) $\Theta(S)$.

The least congruence in the θ-class corresponding to the normal partition P can be characterized as follows (cf. [10]).

LEMMA 1.2. *Let $P = \{E_\alpha : \alpha \in J\}$ be a normal partition of the semilattice of idempotents of an inverse semigroup S. Let $\sigma = \{(a, b) \in S \times S$: there exists an $\alpha \in J$ with $aa^{-1}, bb^{-1} \in E_\alpha$ and $ea = eb$, for some $e \in E_\alpha\}$. Then σ is the smallest congruence on S inducing the partition P.*

By a *one-to-one partial transformation* of a set X, we mean a one-to-one mapping α of a subset Y of X onto a subset $Y' = Y\alpha$ of X. We call Y the *domain* of α, Y' the *range* of α and write $\Delta(\alpha) = Y$, $\nabla(\alpha) = Y'$. If we denote by $\mathscr{I}_X$ the set of all one-to-one partial transformations of X then, with respect to the natural multiplication of mappings, $\mathscr{I}_X$ is an inverse semigroup, called the *symmetric inverse semigroup on* X [2].

Let X be a totally ordered set. By an *ideal* of X, we mean a subset Y of X such that $x \leqslant y$, $y \in Y$ implies that $x \in Y$. By a *principal ideal* we mean an ideal of the form $\{x : x \leqslant y\}$, for some fixed element y. Then we call $\{x : x \leqslant y\}$ the *(principal) ideal generated* by y and denote it by $\langle y \rangle$. Let J_X denote the set of all $\alpha \in \mathscr{I}_X$ such that

(i) $\Delta(\alpha)$ and $\nabla(\alpha)$ are ideals of X;

(ii) α is an order isomorphism of $\Delta(\alpha)$ onto $\nabla(\alpha)$; that is, a one-to-one mapping of $\Delta(\alpha)$ onto $\nabla(\alpha)$ such that, for $x, y \in \Delta(\alpha)$, $x \leqslant y$ if and only if $x\alpha \leqslant y\alpha$.

Then J_X is an inverse subsemigroup of $\mathscr{I}_X$. If T_X denotes the set of α in J_X such that $\Delta(\alpha)$ is a principal ideal then T_X is an inverse subsemigroup of J_X. The semigroup T_X, in the more general situation when X is a semilattice, was introduced and studied by Munn, [6] and [7].

If X is a totally ordered set, $a \in X$, B, and C are subsets of X then "$a < B$" shall mean that $a < b$ for all $b \in B$, and "$B < C$" shall mean that $b < c$, for all $b \in B$, $c \in C$, and similarly with $\leqslant$ replacing $<$. Also $B \backslash C$ shall denote $\{b : b \in B, b \notin C\}$.

For an arbitrary set A, we denote by $| A |$, the *cardinality* of A.

We denote by N the set of nonnegative integers and by Z the set of all integers, under their natural ordering. We denote by N' a set order anti-isomorphic with N.

II. X Totally Ordered

Throughout this section we shall be concerned exclusively with subsemigroups of J_X where X is a totally ordered set.

Let X be a totally ordered set. We call an equivalence relation ρ on X a *convex equivalence* if

(i) $x \leqslant y \leqslant z$, $(x, z) \in \rho$ implies that $(x, y) \in \rho$.

We denote the lattice of convex equivalence relations on X by $E(X)$.

If S is a subsemigroup of J_X, then by a *convex congruence* ρ on X we mean a convex equivalence relation ρ on X such that

(ii) if $x, y \in \Delta(a)$ and $a \in S$ then $(x, y) \in \rho$ if and only if $(xa, ya) \in \rho$.

We observe that, if S is an inverse subsemigroup of J_X then, since we have inverse mappings at our disposal, condition (ii) above on ρ is equivalent to

(ii)′ if $x, y \in \Delta(a)$, $a \in S$ and $(x, y) \in \rho$ then $(xa, ya) \in \rho$.

In the terminology of [10] a convex congruence on X is an s-congruence. Clearly the definition of a convex congruence depends on S; however, since it should not lead to any confusion we generally omit any indication of this dependence. We shall denote the lattice (as it clearly is) of convex congruences on X by $C(X)$ or by $C_S(X)$, when we have occasion to discuss the lattice of convex congruences with respect to distinct semigroups. We denote the maximum and minimum element of $C(X)$, that is, the universal and identity congruences by ω_X and ι_X, respectively. Thus, if X is nontrivial, $C(X)$ has at least two distinct elements. If $C(X)$ has exactly two distinct elements then we say that S is *o-primitive.*

We say that S is *transitive*, or is a *transitive subsemigroup* of J_X, if, for all $x, y \in X$, there exists an $a \in S$ such that $x \in \Delta(a)$ and $xa = y$.

The following lemma is an extension of Holland and McCleary's Theorem 3 in [4], for ordered permutation groups, and may be established in an analogous manner.

Lemma 2.1. *Let X be a totally ordered set and S be a transitive subsemigroup of J_X. Then $C(X)$ is a totally ordered set.*

In the absence of the assumption that S is transitive, $C(X)$ will still be a distributive lattice, as we now demonstrate.

We shall say that a lattice L satisfies the infinite distributive law D if, for any $\rho \in L$ and any subset $\{\sigma_i : i \in I\}$ of L,

$$\rho \wedge \left(\bigvee_I \sigma_i\right) = \bigvee_I (\rho \wedge \sigma_i).$$

LEMMA 2.2 [1, p. 24]. *Let X be a totally ordered set. Then $E(X)$ satisfies the infinite distributive law D.*

PROPOSITION 2.3. *Let X be a totally ordered set and S be a subsemigroup of J_X. Then $C(X)$ is a complete sublattice of $E(X)$ and hence satisfies the infinite distributive law D.*

Proof. Let $A = \{\sigma_i : i \in I\} \subseteq C(X)$. Then clearly the greatest lower bound of A in both $E(X)$ and $C(X)$ is the relation

$$\{(x, y) : (x, y) \in \sigma_i \text{ , for all } i\}.$$

The least upper bound τ of A in $E(X)$ is obtained as follows: $(x, y) \in \tau$ if and only if there exist $x = x_0, x_1, \ldots, x_n = y$ in X such that $(x_{j-1}, x_j) \in \sigma_j$, $j = 1,\ldots, n$, for some $j \in I$. Without loss of generality, let $x \geqslant y$. Then

$$(x, x \wedge x_1) = (x_0 \wedge x_0, x_0 \wedge x_1) \in \sigma_1$$

and

$$(x_0 \wedge x_{j-1}, x_0 \wedge x_j) \in \sigma_j$$

for $j = 2,\ldots, n$. Hence we may assume that $x_j \leqslant x$, for $j = 0,\ldots, n$.

If $x \in \Delta(a)$, for some $a \in S$, then $x_j \in \Delta(a), j = 1,\ldots, n$, and, since $\sigma_j \in C(X)$, for all j, $(x_{j-1}a, x_ja) \in \sigma_j$, for $j = 1,\ldots, n$. Thus $(xa, ya) \in \tau$.

Conversely, suppose that $(x, y) \in \Delta(a)$, $a \in S$ and that $(xa, ya) \in \tau$. Let $xa \geqslant ya$. Then there exist y_k, $k = 0,\ldots, m$, such that $y_0 = xa,\ldots, y_m = ya$, $y_k \leqslant xa$ and, for each k, $(y_{k-1}, y_k) \in \sigma_k$, for some σ_k. Let x_k, $k = 0,\ldots, n$ be such that $x_ka = y_k$ (in particular $x = x_0$, $y = x_n$). Then $(x_{k-1}, x_k) \in \sigma_k$, for $k = 1,\ldots, n$, since $\sigma_k \in C(X)$. Hence $(x, y) \in \tau$. Thus $\tau \in C(X)$ and $C(X)$ is a complete sublattice of $E(X)$.

For any totally ordered set X, let $\bar{X}$ denote the Dedekind completion of X, without end points.

LEMMA 2.4. *Let X be a totally ordered set. For $\alpha \in J_X$ let $\bar{\alpha} \in J_{\bar{X}}$ be such that $\Delta(\bar{\alpha}) = \{\bar{x} \in \bar{X} : \bar{x} \leqslant x_1$, for some $x_1 \in \Delta(\alpha)\}$ and, for $\bar{x} \in \Delta(\bar{\alpha})$, $\bar{x}\bar{\alpha} = \sup\{y\alpha : y \leqslant \bar{x}, y \in X\}$. Then the mapping $\alpha \to \bar{\alpha}$ is an isomorphism of J_X into $J_{\bar{X}}$.*

Proof. Let $\bar{x} \in \Delta(\overline{\alpha\beta})$. Then $\bar{x} \leqslant y$, for some $y \in \Delta(\alpha\beta)$. Thus $y \in \Delta(\alpha)$ and $y\alpha \in \Delta(\beta)$. Also $\bar{x}\bar{\alpha} \leqslant y\bar{\alpha} = y\alpha$. Hence $\bar{x} \in \Delta(\bar{\alpha}\bar{\beta})$.

Conversely, if $\bar{x} \in \Delta(\bar{\alpha}\bar{\beta})$ then $\bar{x} \in \Delta(\bar{\alpha})$ and $\bar{x}\bar{\alpha} \in \Delta(\bar{\beta})$. Thus there exist $u \in \Delta(\alpha)$, $v \in \Delta(\beta)$ such that $\bar{x} \leqslant u$, $\bar{x}\bar{\alpha} \leqslant v$. Let $w = \min\{u\alpha, v\}$, and z be such that $z\alpha = w$. Then $z \leqslant u$, $z\alpha \leqslant v$, and so $z \in \Delta(\alpha)$, and $z\alpha \in \Delta(\beta)$. Thus $z \in \Delta(\alpha\beta)$, $\bar{x} \leqslant z$, and so $\bar{x} \in \Delta(\overline{\alpha\beta})$. Hence $\Delta(\overline{\alpha\beta}) = \Delta(\bar{\alpha}\bar{\beta})$. The remainder of the proof follows routinely.

On account of the natural embedding of the above lemma we henceforth consider J_X as a subsemigroup of $J_{\bar{X}}$.

If A and B are semilattices and α is a mapping of A into B, then α is a *semilattice homomorphism* if, for any $u, v \in A$, $(u \wedge v)\alpha = u\alpha \wedge v\alpha$.

LEMMA 2.5. *Let X be a totally ordered set and S be a subsemigroup of $J_X \subseteq J_{\bar{X}}$. Let α be the mapping of $C(\bar{X})$ into $C(X)$ defined by $\rho\alpha = \rho_{|X \times X}$. Then α is a semilattice homomorphism of $C(\bar{X})$ onto $C(X)$. Also $\rho\alpha = \omega_X$ if and only if $\rho = \omega_{\bar{X}}$; and $\rho\alpha = \iota_X$ if and only if $\rho = \iota_{\bar{X}}$. Thus S is o-primitive on X if and only if S is o-primitive on $\bar{X}$.*

Proof. If $\rho_1, \rho_2 \in C(\bar{X})$ then it is clear that $(\rho_1 \wedge \rho_2)\alpha = \rho_1\alpha \wedge \rho_2\alpha$. Thus, α is a semilattice homomorphism.

Let $\sigma \in C(X)$ and let $\rho = \{(x, y)$: either $x = y$ or there exist $u, v \in X$ with $u \leqslant x$, $y \leqslant v$ and $(u, v) \in \sigma\}$. Clearly ρ is a convex equivalence on $\bar{X}$. Let $(x, y) \in \rho$, $x, y \in \bar{X}$ and $x, y \in \Delta(a)$. There exist $u, v \in X$ such that $u \leqslant x$, $y \leqslant v$ and $(u, v) \in \sigma$. Without loss of generality, let $x \leqslant y$.

Since $y \in \Delta(a)$, there exists an element $w \in X$ such that $y \leqslant w$ and $w \in \Delta(a)$. Let $z = \min\{v, w\}$. Then $(u, z) \in \sigma$, $z \in \Delta(a)$ and $u \leqslant x \leqslant y \leqslant z$. Thus $ua \leqslant xa \leqslant ya \leqslant za$ and $(ua, za) \in \sigma$. Hence $(xa, ya) \in \rho$ and ρ is a convex congruence. Since $\rho\alpha = \sigma$, it follows that α is a semilattice homomorphism of $C(\bar{X})$ onto $C(X)$.

Let $\rho\alpha = \omega_X$. Let $x, y \in \bar{X}$, then there exist $u, v \in X$ such that $u \leqslant x$, $y \leqslant v$. Since $(u, v) \in \omega_X$, we must have $(x, y) \in \rho$. Thus $\rho = \omega_{\bar{X}}$.

Let $\rho\alpha = \iota_X$ and $(x, y) \in \rho$. If $x \neq y$, say $x < y$, then there must exist u, $v \in X$ such that $u \neq v$ and $x \leqslant u$, $v \leqslant y$. Then $(u, v) \in \rho\alpha = \iota_X$, a contradiction. Thus $x = y$ and $\rho = \iota_{\bar{X}}$.

In general, even if S is the group of order preserving permutations of X, the mapping α of Lemma 2.5 need not be a lattice homomorphism, as Example 3 in Section 4 illustrates.

Let Y be a subset of X. We say that Y is *dense* in X if, for all $u, v \in X$ with $u < v$, there exists a $y \in Y$ with $u < y < v$. If, for all $x, y \in Y$ there exists an $a \in S$ such that $x \in \Delta(a)$ and $xa = y$ and if, for $u \in \Delta(a)$, $a \in S$ we have $u \in Y$ if and only if $ua \in Y$, then we call Y a *transitivity class* of X. If S is an inverse semigroup, then clearly, since we have inverse mappings at our disposal, or see Chapter 7 of [2], X is a disjoint union of transitivity classes. If S is not an inverse semigroup, then some elements of X may not belong to a transitivity class.

THEOREM 2.6. *Let X be a totally ordered set such that $|X| > 2$ and let S be an o-primitive inverse subsemigroup of J_X, then either* (i) *X is isomorphic to N' or Z and S is transitive or* (ii) *every transitivity class Y of $\bar{X}$ is dense in X.*

Proof. By Lemma 2.5, S is o-primitive on X if and only if S is o-primitive on $\bar{X}$ and so, without loss of generality, we assume that $X = \bar{X}$.

Since S is an inverse semigroup, X is a disjoint union of transitivity classes.

If X is finite then clearly any convex equivalence relation on X is a convex congruence. Hence, since $|X| > 2$ and S is o-primitive, X is not finite.

Let Y be a trivial transitivity class, that is with just a single element y, say. Define the relation ρ as follows:

$$\begin{aligned}(u, v) \in \rho \Leftrightarrow \text{ either (i) } & u = v = y;\\ \text{or (ii) } & y < u, v;\\ \text{or (iii) } & y > u, v.\end{aligned}$$

Then ρ is clearly a nontrivial convex congruence, contradicting the hypothesis that S is o-primitive. Hence all transitivity classes are nontrivial.

Let Y be any transitivity class. Define the relation σ on X as follows:

$$\begin{aligned}(u, v) \in \sigma \Leftrightarrow \text{ either (i) } & u = v\\ \text{or (ii) } & u < v \quad \text{and} \quad [u, v] \cap Y \subseteq \{u\};\\ \text{or (iii) } & v < u \quad \text{and} \quad [v, u] \cap Y \subseteq \{v\};\end{aligned}$$

where, of course, $[u, v] = \{x \in X : u \leqslant x \leqslant v\}$. Then $\sigma \in C(X)$.

Similarly define the relation τ on X as follows:

$$\begin{aligned}(u, v) \in \tau \Leftrightarrow \text{ either (i) } & u = v;\\ \text{or (ii) } & u < v \quad \text{and} \quad [u, v] \cap Y \subseteq \{v\};\\ \text{or (iii) } & v < u \quad \text{and} \quad [v, u] \cap Y \subseteq \{u\}.\end{aligned}$$

Then $\tau \in C(X)$.

Since σ and τ are both the identity relation when restricted to Y and since S is o-primitive, we must have $\sigma = \tau = \iota_X$.

Hence, either Y is dense or there exist $y_1, y_2 \in Y$ with $y_1 < y_2$ and $[y_1, y_2] = \{y_1, y_2\}$.

Define the relation ν on X as follows:

$$(u, v) \in \nu \Leftrightarrow \text{either (i) } u = v;$$
$$\text{or (ii) } u, v \in Y \text{ and either}$$
$$\text{(a) } u < v, [u, v] \subseteq Y \quad \text{and} \quad |[u, v]| \text{ is finite;}$$
$$\text{or (b) } v < u, [v, u] \subseteq Y \quad \text{and} \quad |[v, u]| \text{ is finite.}$$

Then $\nu \in C(X)$. Since $(y_1, y_2) \in \nu$, $\nu \neq \iota_X$ and hence $\nu = \omega_X$. Thus $X = Y$ and S is transitive. For $x \in X$, if $x > y_1$, then $|[y_1, x]|$ is finite; if $x < y_1$ then $|[x, y_1]|$ is finite. As observed above X must be infinite.

If X has a largest element, then clearly X is isomorphic with N'; otherwise X is isomorphic with Z.

Thus, either X is isomorphic with N' or Z and S is transitive or, for all $u, v \in X$ with $u < v$, there exists a $y \in Y$ with $u < y < v$ that is, Y is dense in X.

COROLLARY 2.7. *Let X be a totally ordered set and let S be an o-primitive subsemigroup of J_X. If there exists a nontrivial transitivity class Y in X then either $X = Y$ and is isomorphic to N' or Z, or Y is dense in X.*

Proof. In Theorem 2.6, the fact that S is an inverse subsemigroup of J_X is only used to establish that every transitivity class is nontrivial. Thereafter, the proof is entirely concerned with an arbitrary (nontrivial) transitivity class, and does not require that S be an inverse semigroup.

If S is a subsemigroup of J_X and Y is a transitivity class of X, then we can define a homomorphism ϕ of S into J_Y, as follows: for $a \in S$ let $\phi_a \in J_Y$ be such that $\Delta(\phi_a) = \Delta(a) \cap Y$ and for $y \in \Delta(\phi_a)$, $y\phi_a = ya$. Then the mapping $\phi : a \to \phi_a$ is the desired homorphism. We call ϕ the *induced representation* of S on Y.

LEMMA 2.8. *Let Y be a transitivity class of X which is dense in X. If either S is a group of order preserving permutations of X or $S \subseteq T_X$, then the homomorphism ϕ of S into J_Y is an isomorphism.*

Proof. Suppose that $S \subseteq T_X$ and $\phi_a = \phi_b$, for $a, b \in S$. Let $\Delta(a) = \langle u \rangle$ and $\Delta(b) = \langle v \rangle$. For $u \neq v$, say $u < v$, then there exists a $y \in Y$ such that $u < y < v$. Then $y \in \Delta(\phi_b) \backslash \Delta(\phi_a)$ and $\phi_a \neq \phi_b$, a contradiction. Hence $u = v$. Now suppose that, for $x \in \Delta(a) = \Delta(b)$, $xa \neq xb$. If $xa < xb$, then there exists an element $y \in Y$ such that $xa < y < xb$. Then $y \in \nabla(b)$, and so there exists a $z \in X$ such that $zb = y$. Thus $z < x$ and $z \in Y$. Since $z < x$, we have $za < xa < y = zb$. Thus $z\phi_a < z\phi_b$, and so $\phi_a \neq \phi_b$, a contradiction. Similarly, if $xb < xa$, we obtain a contradiction. Hence $xa = xb$, for all $x \in \Delta(a) = \Delta(b)$, and so $a = b$. Thus, if $S \subseteq T_X$, we have the lemma. If S is a group of order preserving permutations then the result is clear.

COROLLARY 2.9. *Let X be a totally ordered set, $|X| > 2$ and S be a subsemigroup of J_X. Let S be o-primitive. If S is a group of order preserving permutations or if S is actually contained in T_X, then for any nontrivial transitivity class Y of X the induced representation ϕ of S on Y is an isomorphism of S into J_Y. Moreover, $S\phi$ is transitive and o-primitive on Y.*

Proof. Let Y be a nontrivial transitivity class of X. From Theorem 2.6 and Corollary 2.7, either $Y = X = N'$, or Z and S is transitive on X, or Y is dense in $\bar{X}$. Of course, if S is a group of order preserving permutations of X, then we cannot have $X = N'$. However, if $X = N'$ (and $S \subseteq T_X$) or $X = Z$, then X is the only transitivity class and the induced representation is just the identity isomorphism. Thus, the result holds in this case.

Suppose now that $X \neq N'$ or Z and that Y is any nontrivial transitivity class. By Lemma 2.8, the induced representation ϕ of S on Y is an isomorphism of S into J_Y. First suppose that S is a group of order preserving permutations. Then, since S is o-primitive, X cannot have either a maximum or minimum element. Hence, since Y is dense in $\bar{X}$, we must have $\bar{Y} = \bar{X}$. Let ϕ be the induced representation of S on Y. Then ϕ followed by the natural embedding of J_Y into $J_{\bar{Y}} = J_{\bar{X}}$ is just the natural embedding of S into $J_{\bar{X}}$. Hence by Lemma 2.5, $S\phi$ is o-primitive on Y if and only if $S\phi$ is o-primitive on $\bar{Y}$, that is, if and only if S is o-primitive on $\bar{X}$ which holds, since S is o-primitive on X.

However, if $S \subseteq T_X$ then the isomorphism ϕ of S into J_Y followed by the natural embedding of J_Y into $J_{\bar{Y}} = J_{\bar{X}}$, need not yield the natural embedding of S into $J_{\bar{X}}$, since the domains might not correspond. Therefore, to show that $S\phi$ is o-primitive on Y, we proceed by contradiction.

Suppose that $S\phi$ is not o-primitive on Y. Let ρ be a nontrivial convex congruence on Y. Define σ on $\bar{X}$ by

$$(x, y) \in \sigma \Leftrightarrow \text{either (i) } x = y; \quad \text{or (ii) there exist } u, v \in Y \text{ with } u \leqslant x, y \leqslant v \quad \text{and} \quad (u, v) \in \rho.$$

Then σ is clearly a convex equivalence relation on $\bar{X}$. However, it is possible for σ to fail to be a congruence. Now define the relation τ on $\bar{X}$ by

$$\begin{aligned}(x, y) \in \tau \Leftrightarrow \text{either } & \text{(i) } (x, y) \in \sigma; \\ \text{or } & \text{(ii) } x < y,\ y = \sup x\sigma \quad \text{and} \quad y \notin Y; \\ \text{or } & \text{(iii) } y < x,\ x = \sup y\sigma \quad \text{and} \quad x \notin Y.\end{aligned}$$

Clearly, if $y \notin Y$ and $y = \sup x\sigma$, then $y\sigma = \{y\}$. Hence, the relation τ is a convex equivalence relation on $\bar{X}$ and we wish to show that τ is a congruence.

Let $(x, y) \in \tau$ and $x, y \in \Delta(a)$, $a \in S$. Without loss of generality, let us assume that $x < y$.

First suppose that $(x, y) \in \sigma$. Then, there exist $u, v \in Y$ such that $u \leqslant x < y \leqslant v$ and $(u, v) \in \rho$. If $y \in Y$, then $(u, y) \in \rho$, $ua \leqslant xa < ya$ and $(ua, ya) \in \rho$. Hence $(xa, ya) \in \sigma \subseteq \tau$. Therefore, suppose that $y \notin Y$. Then $ya \notin Y$. If there exists a $z \in Y$ with $(ua, z) \in \rho$ and $ya < z$, then $ua \leqslant xa < ya < z$, and so $(xa, ya) \in \sigma \subseteq \tau$. So, now suppose that $z \in Y$, $(ua, z) \in \rho$ implies that $z < ya$. For any $w \in Y$ such that $xa < w < ya$, we have $w = w'a$ for some $w' \in X$ such that $u \leqslant x < w' < y < v$. Then $(u, w') \in \rho$, $(ua, w) \in \rho$, $(ua, w) \in \sigma$ and $(ua, xa) \in \sigma$. Thus $ya = \sup(ua)\rho = \sup(ua)\sigma = \sup(xa)\sigma$. Since $ya \notin Y$, $(xa, ya) \in \tau$.

Now suppose that $(x, y) \notin \sigma$. Then $y \notin Y$ and $y = \sup x\sigma$. Hence $ya \notin Y$. Since $y = \sup x\sigma$ either $ya = \sup(xa)\sigma$, or $ya \in (xa)\sigma$. In either case $(xa, ya) \in \tau$.

By a very similar argument to the one above, we can show that $(xa, ya) \in \tau$ implies that $(x, y) \in \tau$.

Hence τ is a convex congruence on $\bar{X}$. Since $\rho = \tau_{|Y \times Y}$ and ρ is nontrivial; τ is also nontrivial. Thus S is not o-primitive on $\bar{X}$. But by Lemma 2.5, this contradicts the assumption that S is o-primitive on X. Hence $S\phi$ is o-primitive on Y.

In certain circumstances, the converse of Theorem 2.6 will hold. In particular, when the following condition is satisfied:

(A) If $x, y, z \in X$ and there exists an $a \in S$ such that $x \in \Delta(a)$ and $xa = y$, then there exists an element $b \in S$ such that $x, z \in \Delta(b)$ and $xb = y$.

If S is a group of order preserving permutations of X, then this condition is satisfied.

THEOREM 2.10. *Let X be a totally ordered set and S be an inverse subsemigroup of J_X. If either* (i) *$X = N'$ or Z and S is transitive or* (ii) *every transitivity class of $\bar{X}$ is dense in $\bar{X}$ and S satisfies condition A, then S is o-primitive.*

Proof. Clearly the result is true in case (i). So suppose that (ii) holds. Let ρ be a nontrivial convex congruence on $\bar{X}$. Let $u, v, w \in \bar{X}$ be such that $u < v$, $(u, v) \in \rho$ and $(u, w) \notin \rho$. Suppose that $v < w$, the alternative case $w < u$ being treated similarly.

There exists an $a \in S$ such that $w \in \Delta(a)$ and $u < wa < v$. Since $(ua, wa) \notin \rho$, we must have $(ua)\rho < (wa)\rho = u\rho$. Let $y = \sup(ua)\rho$. Then, for some $b \in S$, $y \in \Delta(b)$ and $u < yb < v$. Thus $(yb, v) \in \rho$. Let $z \in X$ be such that $u = zb$. Then $z < y$ and $(zb, yb) = (u, yb) \in \rho$. Hence $(z, y) \in \rho$ and, since $y = \sup(ua)\rho$, $y \in (ua)\rho$. Since S satisfies condition (A), there exists an element $c \in S$ such that $(yb)c = y$ and $v \in \Delta(c)$. Then $yb < v$ implies that $y = ybc < vc$, and so $vc \notin y\rho = (ua)\rho$. Thus $((yb)c, vc) = (y, vc) \notin \rho$, while $(yb, v) \in \rho$. Thus, we have a contradiction, and S must be o-primitive on $\bar{X}$ and so on X.

Note 1. If, in Theorems 2.6 and 2.10, S is a group of order preserving permutations of X, then these results constitute a slight refinement of a result due to Holland (Theorem 2, [3]) for lattice ordered groups of order preserving permutations of a totally ordered set. In [3], Holland assumes that S is transitive.

Note 2. In Theorem 2.10, we have required that S be an inverse semigroup solely to ensure that X is a disjoint union of transitivity classes. If the hypothesis that S is an inverse semigroup is replaced by the hypothesis that X is a disjoint union of transitivity classes, then the amended theorem will also be true.

If, in Theorem 2.10, we do not assume that S satisfies conditions (A), then the theorem need not be true as the following example illustrates.

EXAMPLE. Let R denote the set of real numbers under their natural ordering. Let $S = \{\alpha \in T_R : \text{for } x \in \Delta(\alpha),\ x \text{ not the maximum element of } \Delta(\alpha),\ x\alpha \in Z \text{ if and only if } x \in Z\}$. Then S is an inverse subsemigroup of T_R. Moreover, S is transitive on R and R is equal to its Dedekind completion. However, S does not satisfy condition (A). If m is any integer, then there exists no element $\alpha \in S$ such that $m + 1 \in \Delta(\alpha)$ and $m\alpha = m + \frac{1}{2}$. Because, if $m + 1 \in \Delta(\alpha)$, then m is not the maximum element of $\Delta(\alpha)$, and so $m\alpha$ is an integer. Thus, S satisfies the conditions of Theorem 2.10 apart from condition (A). Finally, the relation ρ defined on R by

$$(x, y) \in \rho \qquad m < x, y \leqslant m + 1 \qquad \text{for some} \quad m \in Z,$$

is a nontrivial convex congruence on R.

III. APPLICATION TO $\Theta(S)$

In this section we wish to deduce, from the results of the previous section, results regarding $\Theta(S)$ for certain inverse semigroups S.

First, we recall (cf. [2]) that two elements of a semigroup S are said to be $\mathscr{L} - (\mathscr{R})-$) equivalent if they generate the same principal left (right) ideal of S. We write $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$, and call S *bisimple* if it has only one $\mathscr{D}$-class. A necessary and sufficient condition for two idempotents e, f in an inverse semigroup S to be $\mathscr{D}$-equivalent is that there exist an element $a \in S$ such that $aa^{-1} = e$ and $a^{-1}a = f$.

We call a congruence ρ on an inverse semigroup S *idempotent separating*, if no two distinct idempotents of S are ρ-equivalent. Howie [5] has shown that there is a maximum idempotent separating congruence on an inverse semigroup.

We need the following representation of an inverse semigroup due to Munn [7].

LEMMA 3.1. *Let S be an inverse semigroup and $E_S = E$. Define a mapping $\theta : S \to T_E$ by the rule that $a\theta = \theta_a$ where*

(i) $\Delta(\theta_a) = Eaa^{-1}$;

(ii) *for* $e \in \Delta(\theta_a)$, $e\theta_a = a^{-1}ea$.

Then θ is a homomorphism of S into T_E inducing the maximum idempotent separating congruence on S. Moreover, $S\theta$ is transitive on E if and only if S is bisimple.

If the maximum idempotent separating congruence on an inverse semigroup S is the identity congruence, then S is called *fundamental* (cf. [6]). Thus, if S is fundamental, the representation θ of Lemma 3.1 is an isomorphism.

We call an inverse subsemigroup S of T_X *full* if S contains all the idempotents of T_X. It then follows from [6], Theorem 2.6, that a full inverse subsemigroup of T_X is fundamental. In Lemma 3.1, $S\theta$ is a full inverse subsemigroup of T_E.

We also require the following special case of Theorem 6.1 of [10], relating $C(X)$ and $\Theta(S)$;

LEMMA 3.2. *Let X be a totally ordered set and S be a full inverse subsemigroup of T_X. Then $C(X)$ and $\Theta(S)$ are lattice isomorphic.*

THEOREM 3.3. *Let S be an inverse semigroup such that $E = E_S$ is totally ordered with respect to the natural partial ordering. Then $\Theta(S)$ satisfies the infinite distributive law D. If S is also bisimple, then $\Theta(S)$ is totally ordered.*

Proof. Let μ denote the maximum idempotent separating congruence on S. Then, by Lemma 3.1, S/μ is isomorphic with a full inverse subsemigroup of T_E. By Proposition 2.3, $C(E)$ satisfies the infinite distributive law D. Now $\Theta(S) = \Theta(S/\mu)$, and so, by Lemma 3.2, $\Theta(S)$ satisfies the infinite distributive law D.

If, in addition S is a bisimple inverse semigroup, then S/μ acts transitively on E, and so, by Theorem 2.1, $C(E)$ is totally ordered. Hence $\Theta(S) = \Theta(S/\mu)$ is totally ordered.

If S is a transitive inverse subsemigroup of J_X, then by Theorem 2.1, $C(X)$ is totally ordered. However, this need not imply that $\Theta(S)$ is totally ordered as Example 2 in Section 4 illustrates.

We shall require the following result, Proposition 6.3 of [10], in the next theorem.

LEMMA 3.4. *Let X be a totally ordered set and S be a full inverse subsemigroup of J_X. Let $T = S \cap T_X$. Then $C_S(X) = C_T(X)$.*

If S is an inverse semigroup such that E_S is totally ordered, then each $\mathscr{D}$-class of S is a bisimple inverse subsemigroup S by [11]. If S is a bisimple inverse semigroup such that E_S is order isomorphic with $Z(N')$, then we say that S is a bisimple Z-semigroup (bisimple ω-semigroup). Equivalently, let μ be the maximum idempotent separating congruence on S, if S/μ is isomorphic with $T_Z(T_{N'})$, then S is a bisimple Z-semigroup (bisimple ω-semigroup).

THEOREM 3.5. *Let S be an inverse semigroup such that $|E_S| > 2$, E_S is totally ordered and $|\Theta(S)| = 2$. Then either $E_S = N'$ (and S is a bisimple ω-semigroup) or $E_S = Z$ (and S is a bisimple Z-semigroup) or each class of $\mathscr{D}$-equivalent idempotents is dense in E_S. Moreover, for each $\mathscr{D}$-class D of S, D is a bisimple inverse semigroup such that E_D is totally ordered and $|\Theta(D)| = 2$. If S is fundamental, then so also is each $\mathscr{D}$-class D.*

Proof. Let μ be the maximum idempotent separating congruence on S and $E = E_S$. Then S/μ is (isomorphic to) a full inverse subsemigroup of T_E. Since $|\Theta(S)| = 2$, we have $|\Theta(S/\mu)| = 2$ and, by Lemma 3.2, $|C(E)| = 2$. Hence, by Theorem 2.10, either $E = N'$ or Z and S/μ is transitive, or every transitivity class in E is dense in E.

If $E = N'$, then S/μ is a transitive inverse subsemigroup of $T_{N'}$. Thus $S/\mu = T_{N'}$ and S is a bisimple ω-semigroup. Similarly, if $E = Z$, then S is a bisimple Z-semigroup.

Now suppose that $E \neq N'$ or Z and that every transitivity class is dense in E. Let $e, f \in E$ be elements in the same transitivity class. Then, in the notation of Lemma 3.1, there exists an $a \in S$, such that $e \in \Delta(\theta_a)$, and $e\theta_a = f$. Now $e \in \Delta(\theta_e)$ and $e\theta_e = e$. Hence $e \in \Delta(\theta_e\theta_a) = \Delta(\theta_{ea})$. Hence $e \leqslant (ea)(ea)^{-1} = eaa^{-1}e = eaa^{-1} \leqslant e$; that is, $(ea)(ea)^{-1} = e$. Also, $f = e\theta_{ea} = (ea)^{-1}e(ea) = a^{-1}ea = (ea)^{-1}(ea)$. Thus, if $b = ea$, then $bb^{-1} = e$ and $b^{-1}b = f$. Hence $(e, f) \in \mathscr{D}$.

Conversely, if $(e, f) \in \mathscr{D}$ then, for some $a \in S$, $aa^{-1} = e$ and $a^{-1}a = f$. Then $e\theta_a = f$ and e and f belong to the same transitivity class. Thus, the transitivity classes of E are just the classes of $\mathscr{D}$-equivalent idempotents in E. Thus, any class of $\mathscr{D}$-equivalent idempotents is dense in E.

To establish the second statement, only the assertion that $|\Theta(D)| = 2$, remains to be shown.

If S is a bisimple ω-semigroup or a bisimple Z-semigroup, then the assertion is clear. (For details of this and other results on bisimple ω-semigroups and bisimple Z-semigroups see [8] and [12].) Suppose that neither of these possibilities holds, and that D is a $\mathscr{D}$-class with $E_D = Y$. Then Y is a transitivity class in the representation of S/μ on E, and so by Corollary 2.9, the induced representation ϕ of S/μ on Y is faithful. Clearly $(S/\mu)\phi \cap T_Y = (D/\mu)\phi$ and $D\phi$ is full in T_Y. Thus $\Theta(D) = \Theta(D/\mu) = \Theta((D/\mu)\phi)$ is isomorphic with $C_{(D/\mu)\phi}(Y)$. However, by Lemma 3.4,

$$C_{(S/\mu)\phi}(Y) = C_{(D/\mu)\phi}(Y).$$

So it only remains to be shown that $C_{(S/\mu)\phi}(Y)$ is of cardinality 2. But this follows from Corollary 2.9. Hence $|\Theta(D)| = 2$.

Now, suppose that S is fundamental. If S is a bisimple ω-semigroup or a bisimple Z-semigroup, then S has only one $\mathscr{D}$-class and, by assumption, it is fundamental. So let us assume that each $\mathscr{D}$-class D of S is such that E_D is dense in E_S and let D be an arbitrary $\mathscr{D}$-class. Since μ is just the identity congruence, we have, by Corollary 2.9, that the induced representation ϕ of S on E_D (induced from the representation of S from Lemma 3.1 on E_S) is faithful and clearly maps D onto a full inverse subsemigroup of T_{E_D}. Hence D is fundamental.

In connection with Theorem 3.5, Munn has shown independently (unpublished) that if S is an inverse semigroup such that E_S is totally ordered, $|E_S| > 2$, and S is congruence free, that is, admits only the identity and universal congruences, then E_S is dense in itself. Now if S is congruence free and $|E_S| > 2$, then necessarily $|\Theta(S)| = 2$. Also, since the additive group of integers is a homomorphic image of any bisimple ω-semigroup or bisimple Z-semigroup, such a semigroup cannot be congruence free. Hence we could deduce Munn's result from Theorem 3.5.

If in Theorem 3.5, S is neither a bisimple Z-semigroup, nor a bisimple ω-semigroup then, even although E_D is dense in E_S, for each $\mathscr{D}$-class D, the $\mathscr{D}$-classes need not be isomorphic semigroups. Let $Q(R)$ denote the set of rational (real) numbers under their natural ordering. Let $S = \{\alpha \in T_R : \text{for } x \in \Delta(\alpha),\ x\alpha \in Q \text{ if and only if } x \in Q\}$. Then R has two transitivity classes, namely Q and $R\backslash Q$. Now E_S is isomorphic to R and, for $e, f \in E_S$ such that $\Delta(e) = \langle x \rangle$ and $\Delta(f) = \langle y \rangle$, it follows that $(e, f) \in \mathscr{D}$ if and only if x and y belong to the same transitivity class. Thus S has two $\mathscr{D}$-classes C and D such that E_C is isomorphic to Q and E_D is isomorphic to $R\backslash Q$. Hence C and D are certainly not isomorphic.

The converse of the first part of Theorem 3.5 does not hold, as it stands. In other words, from the fact that S is an inverse semigroup such that each class of $\mathscr{D}$-equivalent idempotents of S is dense in E_S, it does not follow that $|\Theta(S)| = 2$. Let $X = R \backslash Z$, where R is the set of real numbers and Z is the set of integers. Let $S = T_X$. Then X is dense in X and S is transitive on X. However, the relation ρ defined on X by $(x, y) \in \rho \Leftrightarrow m < x, y < m + 1$, for some $m \in Z$, is a nontrivial convex congruence on X. Hence $|C(X)| > 2$ and so $|\Theta(S)| > 2$. (See also the example at the end of Section 2.)

Thus it appears essential, for the converse, to consider the representation of S on $\bar{E}_S$, via Lemma 3.1 and Lemma 2.4. As a corollary of Theorem 2.10, one can then obtain a rather inelegant converse to Theorem 3.5, which we decline to state.

We conclude this section by identifying a certain sublattice of $\Lambda(S)$, where S is an inverse semigroup such that E_S is totally ordered.

PROPOSITION 3.6. *Let S be an inverse semigroup such that E_S is totally ordered. Then the set $\Sigma(S)$ consisting of the minimum congruences from each θ-class is a sublattice of $\Lambda(S)$. Moreover $\Sigma(S)$ is isomorphic to $\Theta(S)$ and so is a distributive lattice.*

Proof. Let $\sigma_1, \sigma_2 \in \Sigma(S)$ and σ_3 be the minimum element in the θ-class containing $\sigma_1 \vee \sigma_2$. Then, by the choice of σ_3, $\sigma_3 \subseteq \sigma_1 \vee \sigma_2$. But $\sigma_1, \sigma_2 \subseteq \sigma_3$ implies that $\sigma_1 \vee \sigma_2 \subseteq \sigma_3$. Thus $\sigma_1 \vee \sigma_2 = \sigma_3 \in \Sigma(S)$.

Now, let σ_3 be the minimum congruence in the θ-class containing $\sigma_1 \wedge \sigma_2$. Then $\sigma_3 \subseteq \sigma_1 \wedge \sigma_2$, and we want the converse inclusion. Let $(a, b) \in \sigma_1 \wedge \sigma_2$. By Lemma 1.2, we have $(aa^{-1}, bb^{-1}) \in \sigma_1 \wedge \sigma_2$ and so $(aa^{-1}, bb^{-1}) \in \sigma_3$. Also, by Lemma 1.2, there exist idempotents e, f such that $(e, aa^{-1}) \in \sigma_1$, $ea = eb$, $(f, aa^{-1}) \in \sigma_2$ and $fa = fb$. Without loss of generality, we may assume that $e, f \leqslant aa^{-1}, bb^{-1}$. Since E_S is totally ordered, e and f are comparable. So suppose that $e \leqslant f$. Then $e \leqslant f \leqslant aa^{-1}, bb^{-1}$. Hence $(f, aa^{-1}) \in \sigma_1 \wedge \sigma_2$, and so $(f, aa^{-1}) \in \sigma_3$. Moreover, $fa = fb$. Thus, by Lemma 1.2, $(a, b) \in \sigma_3$ and $\sigma_1 \wedge \sigma_2 = \sigma_3 \in \Sigma(S)$. Hence $\Sigma(S)$ is a sublattice of $\Lambda(S)$.

Since $\Sigma(S)$ contains exactly one element from each θ-class, $\Sigma(S)$ is isomorphic to $\Theta(S)$ and hence is a distributive lattice by Theorem 3.3.

For an example to illustrate that $\Sigma(S)$ need not be a sublattice of $\Lambda(S)$ for an arbitrary inverse semigroup S, see [9], Section 6, Example 2.

IV. EXAMPLES

This section is devoted to examples. One of these demonstrates the existence of a bisimple inverse semigroup S such that $\Theta(S)$ is an arbitrary finite chain and each θ-class contains only a single element. Munn, in his discussion on congruence free inverse semigroups in [6], has provided such examples where $|\Theta(S)| = 2$.

We need the following result from [10], by combining Propositions 5.1, 5.2, and Theorem 6.1.

PROPOSITION 4.1. *Let X be a totally ordered set and S be a full inverse subsemigroup of T_X. Then $\Theta(S)$ is isomorphic with $C(X)$. Let $\rho \in C(X)$. Define $\xi = \xi_\rho$ on S by*

$$\begin{aligned}(a, b) \in \xi \Leftrightarrow & \text{ (i) } U(a) = U(b);\\ & \textit{and } \text{(ii) } x \in \Delta(a), y \in \Delta(b) \quad \textit{and} \quad (x, y) \in \rho\\ & \qquad \textit{implies that } (xa, yb) \in \rho,\end{aligned}$$

where, for $\alpha \in T_X$, $U(\alpha) = \{x\rho : x\rho \cap \Delta(\alpha) \neq \phi\}$. Define $\eta = \eta_\rho$ on S by

$$\begin{aligned}(a, b) \in \eta \Leftrightarrow & \text{ (i) } U(a) = U(b),\\ & \textit{and } \text{(ii) } \textit{If } x\rho \in U(a) = U(b), \textit{ then there exists a}\\ & \qquad y \in x\rho \cap \Delta(a) \cap \Delta(b) \textit{ such that}\\ & \qquad za = zb, \quad \textit{for all} \quad z \leqslant y, z \in X.\end{aligned}$$

Then ξ and η are, respectively, the maximum and minimum congruences on S in the θ-class corresponding to ρ.

Let n be a positive integer. We now construct a bisimple inverse semigroup S such that $|\Theta(S)| = n + 1$, E_S is totally ordered and each θ-class contains only one element.

For an inverse semigroup T with idempotents E, we write $E\omega = \{a \in T : e \leqslant a$, in the natural partial ordering of T, for some $e \in E\} = \{a \in T : ea^{-1} = e$, for some $e \in E\}$.

EXAMPLE 1. Let $X = R_1 \times R_2 \times \cdots \times R_n$ where $R_i = R$ (the set of real numbers), $i = 1,\ldots, n$.

Order X lexicographically by defining.

$$(x_1, \ldots, x_n) > (y_1, \ldots, y_n) \Leftrightarrow x_i > y_i,$$

where i is the largest integer for which $x_i \neq y_i$. Let E_i denote the set of idempotents of T_{R_i}.

Let S denote the set of elements α of T_X such that, if $\Delta(\alpha) = \langle (y_1, \ldots, y_n)\rangle$ and $(x_1, \ldots, x_n) \in \Delta(\alpha)$ then

$$(x_1, \ldots, x_n)\alpha = (x_1\alpha(x_2, \ldots, x_n), x_2\alpha(x_3, \ldots, x_n), \ldots, x_n\alpha_n),$$

where

(i) $\alpha_n \in E_n\omega$;

(ii) $\alpha(x_{i+1}, \ldots, x_n) \in E_i\omega$ with domain $\langle y_i \rangle$, if $x_{i+1} = y_{i+1}, \ldots, x_n = y_n$;

and $\alpha(x_{i+1}, \ldots, x_n)$ is the identity permutation of R_i otherwise.

Then S is a full transitive inverse subsemigroup of T_X and, hence E_S is totally ordered and S is bisimple.

For each $i = 1, 2, \ldots, n$ define the relation ρ_i on X by

$$((x_1, \ldots, x_n), (x_1', \ldots, x_n')) \in \rho_i \Leftrightarrow x_i = x_i', x_{i+1} = x_{i+1}', \ldots, x_n = x_n'.$$

Then each ρ_i is easily seen to be a convex congruence on X and, moreover, if ρ_{n+1} is the universal congruence, then $C(X) = \{\rho_1, \ldots, \rho_{n+1}\}$ where $\rho_1 \subset \rho_2 \subset \cdots \rho_n \subset \rho_{n+1}$. Thus, $C(X)$ is an $n + 1$ element chain with ρ_1 the identity congruence and ρ_{n+1} the universal congruence.

Now let ρ_i, $1 < i \leqslant n + 1$, induce the congruences ξ_i and η_i as in Proposition 4.1. We wish to show that $\xi_i = \eta_i$.

Let $(a, b) \in \xi_i$. Let $u\rho_i \cap \Delta(a) \neq \phi$. Then, $u\rho_i \cap \Delta(b) \neq \phi$. Let $u = (u_1, \ldots, u_n)$, $\Delta(a) = \langle(x_1, \ldots, x_n)\rangle$ and $\Delta(b) = \langle(y_1, \ldots, y_n)\rangle$. Choose $z_{i-1} \in R_{i-1}$ such that

(a) $z_{i-1} < x_{i-1}, y_{i-1}$;

(b) $y\alpha(u_i, \ldots, u_n) = y$, for all $y \in R_{i-1}, y \leqslant z_{i-1}$;

(c) $y\beta(u_i, \ldots, u_n) = y$, for all $y \in R_{i-1}, y \leqslant z_{i-1}$.

Choose $z_1, \ldots, z_{i-2}$ arbitrarily in $R_1, \ldots, R_{i-2}$, respectively. Let $z = (z_1, \ldots, z_{i-1}, u_i, \ldots, u_n)$. Then $(z, u) \in \rho_i$ and, for $z' \leqslant z$, $z'a = z'b$. Thus $(a, b) \in \eta_i$ and $\xi_i = \eta_i$. Thus the θ-classes corresponding to $\rho_2, \ldots, \rho_{n+1}$ have only one element. Since S is full in T_X, S is fundamental and so the θ-class corresponding to ρ_1 has only the one element. Thus S has the desired properties.

Note that in the above example each R_i could be replaced by any totally ordered field, with possibly distinct fields replacing distinct R_i.

Using the techniques of [4], examples similar to the above can be constructed to yield a more general totally ordered set for $\Theta(S)$.

We now give an example to show that it does not follow from the fact that S is a transitive inverse subsemigroup of J_X that $\Theta(S)$ is totally ordered. (cf. Theorems 2.1 and 3.3).

EXAMPLE 2. Let $X = Z \times Z$, where $(m, n) > (r, s)$ if and only if $m > r$ or $m = r$ and $n > s$. Let $S = \{\alpha \in J_X : \Delta(\alpha)$ is a proper ideal of $X\}$. For $x \in X$, let e_x denote the idempotent of S with $\Delta(e_x) = \langle x \rangle$. For $m \in Z$, let e_m denote the idempotent of S with $\Delta(e_m) = \{(r, s) : r \leqslant m\}$. Then

$E_S = \{e_x : x \in X\} \cup \{e_m : m \in Z\}$, and S is transitive on X. Define the relations σ and τ on E_S by

$$
\begin{aligned}
(e_u, e_v) \in \sigma \Leftrightarrow \text{ either (i) } & u = (m, n),\ v = (m, r), && \text{for some } m \in Z;\\
\text{or (ii) } & u = (m, n),\ v = m, && \text{for some } m \in Z;\\
\text{or (iii) } & u = m,\ v = (m, n), && \text{for some } m \in Z;
\end{aligned}
$$

and

$$
\begin{aligned}
(e_u, e_v) \in \tau \Leftrightarrow \text{ either (i) } & u = (m, n),\ v = (m, r), && \text{for some } m \in Z;\\
\text{or (ii) } & u = (m, n),\ v = m - 1, && \text{for some } m \in Z;\\
\text{or (iii) } & u = m - 1,\ v = (m, n), && \text{for some } m \in Z.
\end{aligned}
$$

Then σ and τ are noncomparable normal equivalences on E_S. Thus, $\Theta(S)$ is not totally ordered.

We finally give an example to show that the semilattice homomorphism of Lemma 2.5, need not be a lattice homomorphism.

EXAMPLE 3. Let X be as in Example 2, and S be the group of order preserving permutations of X.

Then $\bar{X} = X \cup \{x_m : m \in Z$, where $x_m > (r, s)$ if and only if $r \leqslant m$. Define the relations π, ρ on $\bar{X}$ by

$$
\begin{aligned}
(u, v) \in \pi \Leftrightarrow \text{ either (i) } & u = (m, n),\ v = (m, r), && \text{for some } m \in Z;\\
\text{or (ii) } & u = (m, n),\ v = x_m, && \text{for some } m \in Z;\\
\text{or (iii) } & u = x_m,\ v = (m, n), && \text{for some } m \in Z;
\end{aligned}
$$

and

$$
\begin{aligned}
(u, v) \in \rho \Leftrightarrow \text{ either (i) } & u = (m, n),\ v = (m, r), && \text{for some } m \in Z;\\
\text{or (ii) } & u = (m, n),\ v = x_{m-1}, && \text{for some } m \in Z;\\
\text{or (iii) } & u = x_{m-1},\ v = (m, n), && \text{for some } m \in Z.
\end{aligned}
$$

Then $\pi, \rho \in C(\bar{X})$ and $\pi \vee \rho = \omega_{\bar{X}}$. Let λ be the relation defined on X by $(u, v) \in \lambda \Leftrightarrow u = (m, n),\ v = (m, r)$, for some $m \in Z$. Then $\lambda \in C(X)$, and $\pi\alpha = \rho\alpha = \lambda$. Thus, $\pi\alpha \vee \rho\alpha = \lambda \neq \omega_X = \omega_{\bar{X}}\alpha = (\pi \vee \rho)\alpha$. Hence, α is not a lattice homomorphism.

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